The \textit{Riemann zeta function} is defined by
\[ \zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s}, \quad \text{Re}(s) > 1. \]

It can be extended to a meromorphic function on the whole complex plane, and this meromorphic extension satisfies a functional equation. The famous Riemann hypothesis predicts that all the zeros of \( \zeta \) inside the critical strip \( 0 < \text{Re}(s) < 1 \) in fact have real part \( \frac{1}{2} \). The Riemann zeta function can also be written as the infinite product
\[ \zeta(s) = \prod_p \frac{1}{1 - p^{-s}} \]
where the product runs over all prime numbers. In this form it can be generalized to any scheme \( X \) of finite type over \( \mathbb{Z} \):
\[ \zeta_X(s) := \prod_{x \in |X|} \frac{1}{1 - (\#k(x))^{-s}} \]

Here the index set \( |X| \) consists of the closed points of \( X \) and \( k(x) \) denotes the residue field of \( x \in |X| \). The product converges for \( \text{Re}(s) > \dim X \). This so called \textit{Hasse–Weil zeta function} is defined in particular if \( X \) is a smooth projective variety over the finite field \( \mathbb{F}_q \) with \( q \) elements. In this setting, the analog of the Riemann hypothesis was proven by Deligne [Del74]. A key ingredient in his proof is that the Hasse–Weil zeta function has the form
\[ \zeta_X(s) = Z_X(q^{-s}) \]
where \( Z_X \) is the rational function
\[ Z_X(t) = \prod_{i=0}^{2\dim X} P_i(t)^{(-1)^{i+1}} \]
and \( P_i \) is the characteristic polynomial of the Frobenius acting on the \( i \)-th ℓ-adic cohomology group of \( X \). For the Riemann zeta function in turn, an expression like (1) cannot exist. Instead, the idea is to write the zeta function \( \zeta(s) \) (or more generally \( \zeta_X(s) \) for \( X \) of finite type over \( \mathbb{Z} \)) itself as an alternating product of characteristic ‘polynomials’ of some operator on yet to find cohomology groups. As \( \zeta(s) \) has infinitely many zeros, these cohomology groups have to be infinite dimensional, the ‘polynomials’ have to be power series and the determinant in their definition has to be a \textit{regularized determinant}. One of Deninger’s main results and goal of our seminar is that the Hasse–Weil zeta function \( \zeta_X(s) \) can in fact be written as alternating product of characteristic power series of an operator acting on some infinite dimensional \( \mathbb{C} \)-vector space.
Motivated by the properties of the Selberg zeta function, Deninger has also developed a quite consistent picture how the expected cohomology theory should behave. In particular he observed similarities with the cohomology of dynamical systems on foliated spaces. This indicates that the desired cohomology could appear as the cohomology of some foliated space associated with the scheme $X$ equipped with a Frobenius flow. In the last talks of the seminar, we will see how Deninger’s hypothetical cohomology theory would give a proof of the Riemann hypothesis, and how one may construct such a foliated space with Frobenius flow for an elliptic curve over a finite field.

1. Seminar schedule

**Talk 1: Introduction and Overview.**

**Talk 2+3: Regularized determinants I+II.** Introduce regularized determinants, dimensions and super-dimensions. Discuss the basic properties of regularized determinants following [Den94, §1]. Restate the classical $\Gamma$-function in terms of regularized determinants [Den91, §2].

Main references: [Den94, §1], [Den91, §2].

**Talk 4+5: The Selberg zeta function.**

**Talk 6+7: Riemann-Hilbert correspondence I+II.** The aim of this talk is to explain the Riemann–Hilbert correspondence for $\mathbb{G}_m$ following [Den94, §2]. It might make sense to recall the statement (without proof) of the general Riemann-Hilbert correspondence (see e.g. [BGK+87, Ch. III by Haefliger or Ch. IV by Malgrange]). Then explain the Riemann-Hilbert correspondence for $\mathbb{G}_m$ over $\mathbb{C}$ and the applications to regularized determinants following [Den94, §2].

Main references: [Den94, §2].

**Talk 8: Non-Archimedean local $L$-factors.** Briefly recall the definition of mixed motives in the sense of [Jan90] and [Del89]. For us, it is enough to think about a mixed motive as a tuple of certain cohomology groups (plus extra structure) associated to a smooth variety over a field of characteristic zero. The aim of this talk is to present the construction of the local $L$-factors at non-Archimedean places in terms of regularized determinants following [Den94, §3].

Main reference: [Den94, §3].

**Talk 9: Logarithmic connections and filtered vector spaces.** As a prerequisite for the construction of the Archimedean local $L$-factors, we have to discuss Logarithmic connections and their associated filtered vector spaces following [Den94, §5]. Introduce logarithmic connections on $\mathbb{G}_a$. Discuss the functor associating a logarithmic connection to a filtered vector space and its properties. Finally, relate generalized determinants of certain logarithmic connections to products of $\Gamma$-functions.

Main reference: [Den94, §5].
Talk 10: Archimedean local $L$-factors. Briefly recall the definition of mixed Hodge structures [Del71, SS2.1, 2.3] or [PS08]. Recall the Hodge decomposition and explain the pure Hodge structure associated to smooth and proper varieties over $\mathbb{C}$ [Del71, §2.2]. Recall the classical definition of the Archimedean local $L$-factors following Serre [Ser70]. Explain how to express the local Archimedean $L$-factors in terms of regularized determinants following [Den94, §6].

Main reference: [Den94, §6].

Talk 11: The Riemann hypothesis. Following [Den98, §3] and [Den94, §7] explain how the Riemann hypothesis would follow from a suitable ’arithmetic cohomology theory’ with certain properties. In particular, explain the expected resulting formula for the Riemann $\xi$-function as regularized product and prove it if time permits [Den92, §4].

Main reference: [Den98, §3], [Den94, §7].

Talk 12: Foliated space for an elliptic curve. Deninger observed analogies between the expected cohomology theories and dynamical systems of foliated spaces [Den98]. The aim of the following two talks is to construct such a foliated space which gives the zeta function of an elliptic curve over a finite field. In this talk, explain the construction of the associated laminated foliated space of an elliptic curve over a finite field following [Lei05, §2.2.]. The reference [Den02] might also be helpful.

Main reference: [Lei05, §2.2.], [Den02, §3].

Talk 13: The zeta function via $L^2$-cohomology. Explain how the zeta function of an elliptic curve arises as a regularized characteristic power series of the infinitesimal generator of the Frobenius flow on the leafwise $L^2$-cohomology of the laminated foliated space constructed in the previous talk.

Main reference: [Lei05, §2.3.], [Den02, §3].

References


