Ergodic theory and integral approximations of Gromov's simplicial volume

Regensburg, 27 September 2017

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2. The simplicial volume $\|M\|$.
3. Variations of the simplicial volume.

Simplicial volume(s)

M compact connected oriented n-manifold, $R = \mathbb{Z}, \mathbb{R}$.

ℓ^1-norm on singular chains:

$$\alpha = \sum_{j=1}^{k} a_j \sigma_j \ \in C_i(M, R) \qquad \|\alpha\| = \sum_{j=1}^{k} |a_j|$$

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Taking the infimum over representatives, we have a norm on $Z_i(M, R)$ and a seminorm on $H_i(M, R)$.

If $[M]_{\mathbb{Z}} \in H_n(M, \mathbb{Z})$ is the integral fundamental class of M, then the integral simplicial volume of M is

$$\|M\|_{\mathbb{Z}} = \|[M]_{\mathbb{Z}}\| \quad (> 0)$$

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Of course

$$\|M\| \leq \|M\|_{\mathbb{Z}}$$

Behaviour with respect to coverings

$f\colon N \to M$ of degree d. If z_N is a fundamental cycle of N, then

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If f is a covering, cycles in M can be pulled back to N, and

$$\|M\| = \frac{\|N\|}{|d|}$$

i.e. the simplicial volume is a characteristic number.

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$$\|S^n\| = \|S^n \times N\| = 0 \qquad \forall n \geq 1$$

Some classical results

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Let M be endowed with a Riemannian structure.

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Theorem (Gromov)

If $\pi_1(M)$ is amenable, then $\|M\| = 0$.

Stable integral simplicial volume

We promote integral simpicial volume to a characteristic number by defining the stable integral simplicial volume as follows:

$$\|M\|_{\mathbb{Z}}^{\infty} = \inf_{\widetilde{M} \xrightarrow{d} M} \left\{ \frac{\|\widetilde{M}\|_{\mathbb{Z}}}{d} \right\}$$

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Stable integral simplicial volume vs. simplicial volume

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For which manifolds do we have $\|M\| = \|M\|_{\mathbb{Z}}^{\infty}$*?*

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Question

If M *is aspherical and* $\pi_1(M)$ *is residually finite, then is it true that* $\|M\| = \|M\|_{\mathbb{Z}}^{\infty}$*?*

A conjecture by Gromov

Conjecture (Gromov)

Let M be aspherical with $\|M\| = 0$. Then $\chi(M) = 0$.

Proposition

$$\|M\|_{\mathbb{Z}}^{\infty} = 0 \implies \chi(M) = 0$$

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Gromov's conjecture would be proved if

$$\|M\| = 0 \quad \implies \quad \|M\|_{\mathbb{Z}}^{\infty} = 0$$

for every aspherical M.

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$$\|M\|_{\mathbb{Z}}^{\infty} = 0 \implies \chi(M) = 0$$

Some (negative) results

Theorem (F.–Francaviglia–Martelli)

Let $n \geq 4$. There exists $k_n > 1$ s.t. for every hyperbolic n-manifold M

$$\frac{\|M\|_{\mathbb{Z}}^{\infty}}{\|M\|} \geq k_n > 1$$

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Theorem (Löh–Pagliantini)

There exists a sequence M_i of hyp. 3-manifolds s.t.

$$\frac{\|M_i\|_{\mathbb{Z}}^{\infty}}{\|M_i\|} \to 1 \ .$$

The simplicial volume of hyperbolic manifolds

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Let v_n be the volume of the regular ideal simplex in $\mathbb{H}^n$.

Theorem (Haagerup, Munkholm)

If Δ is any geodesic simplex in $\overline{\mathbb{H}}^n$, then

$$\mathrm{Vol}(\Delta) \leq v_n \ ,$$

and the equality holds if and only if Δ is regular ideal.

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and

$$\sum |a_i| \geq \frac{\mathrm{Vol}(M)}{v_n}, \qquad \|M\| \geq \frac{\mathrm{Vol}(M)}{v_n}$$

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Proving the converse inequality boils down to constructing real fundamental cycles supported on a set of straight simplices with close-to-maximal volume.

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This proves that

$$\|M\|_{\mathbb{Z}}^{\infty} > \|M\|$$

in dimension ≥ 4.

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so we may define

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The boundary operator $\mathrm{Id} \otimes \partial_*$ defines the homology

$$H_*(M, X)$$

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$$[M]_{\mathbb{Z}} \to [M]_X \to [M]$$

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$C_*(M, X)$ is endowed with the norm

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$$\|M\|_X = \|[M]_X\|$$

Integral foliated simplicial volume

Definition (Gromov)

The integral foliated simplicial volume of M is

$$\|M\|_{\mathcal{F}} = \inf_X \|M\|_X$$

Since the maps

$$H_*(M, \mathbb{Z}) \to H_*(M, X) \to H_*(M, \mathbb{R})$$

are norm-non-increasing,

$$\|M\|_{\mathbb{Z}} \geq \|M\|_{\mathcal{F}} \geq \|M\|$$

Why foliated?

The diagonal action of Γ on $\widetilde{M} \times X$ defines the flat bundle

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Being flat, E admits a foliation transverse to the fibers.

An X-fundamental cycle may be seen as a collection of locally finite integral fundamental cycles on the leaves.

Why do we care?

Theorem (Gromov, Schmidt)

If $\|M\|_{\mathcal{F}} = 0$, then the ℓ^2-Betti numbers of M vanish. In particular,

$$\|M\|_{\mathcal{F}} = 0 \quad \Longrightarrow \quad \chi(M) = 0$$

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Therefore, Gromov's conjecture may be reduced to

Conjecture

If M is aspherical, then

$$\|M\| = 0 \implies \|M\|_{\mathcal{F}} = 0$$

Examples

If $H \triangleleft \Gamma$ is of finite index d with associated covering $\overline{M}$, then

$$\Gamma \curvearrowright \Gamma/H\,, \qquad E = \overline{M}\,, \qquad \|M\|_{\Gamma/H} = \frac{\|\overline{M}\|_{\mathbb{Z}}}{d}$$

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Theorem (Löh-Pagliantini)

$$\|M\|_{\widehat{\Gamma}} = \|M\|_{\mathbb{Z}}^{\infty}$$

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Theorem (F.–Löh–Sauer–Pagliantini)

Let $n \geq 4$. Then there exists $k_n > 1$ such that

$$\|M\|_{\mathcal{F}} \geq k_n \|M\| > \|M\|$$

for every closed hyperbolic n-manifold M.

More on hyperbolic manifolds

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Hyperbolic 3-manifolds

Recall that there exists a sequence of hyerbolic 3-manifolds M_i s.t.

$$1 \leq \frac{\|M_i\|_{\mathcal{F}}}{\|M_i\|} \leq \frac{\|M_i\|_{\mathbb{Z}}^{\infty}}{\|M_i\|} \to 1$$

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Corollary (Löh–Pagliantini)

If M is a hyperbolic 3-manifold, then

$$\|M\|_{\mathcal{F}} = \|M\|$$

Comparing parameter spaces

Definition

Let X, Y be probability Γ-spaces. Then X is weakly contained in Y if, roughly speaking, the action of every finite subset of Γ on any finite configuration of measurable sets in X can be realized also in Y up to an arbitrarily small error.

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Theorem (F.–Löh–Sauer–Pagliantini)

Suppose that X is weakly contained in Y. Then

$$\|M\|_Y \leq \|M\|_X$$

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Corollary

If $\pi_1(M)$ has *EMD**, then

$$\|M\|_{\mathcal{F}} = \|M\|_{\mathbb{Z}}^{\infty}$$

Back to hyperbolic 3-manifolds

Theorem (Kechris, Bowen, Tucker-Drob, Agol)

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