Obstructions to Anosov Diffeomorphisms

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Overview

1. Background
   - Basic definitions
   - Smale's Question
   - Previous results
2. Main Theorem
   - Counting periodic points
   - Sketch of proof
3. Main Application

Anosov diffeomorphisms

A diffeomorphism $f : M \to M$ of a closed smooth manifold M is *Anosov* if TM splits as a direct sum $TM = E^u \oplus E^s$ of df-invariant subbundles, and df is expansive on E^u and contractive on E^s, i.e. there is a $C > 0$ and $\lambda > 1$ with

- $||df^m(u)|| \geq C\lambda^m ||u||$ for all $u \in E^u$, and
- $||df^m(u)|| \leq C\lambda^{-m} ||u||$ for all $u \in E^s$.

Example

Consider the linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by a hyperbolic matrix in $GL_2(\mathbb{Z})$. Then this descends to an Anosov diffeomorphism $f : T^2 \to T^2$. E^u, E^s are given by the subspaces with slope corresponding to the eigenspaces of the matrix.

Smale's Question

More generally, one can start with G a simply-connected nilpotent Lie group, $\Gamma \leq G$ a torsion-free uniform lattice, and $\phi : G \to G$ an automorphism leaving Γ invariant. Then ϕ descends to a diffeomorphism $f : M \to M$ where $M = G/\Gamma$. With appropriate choices (e.g. $d\phi : \mathfrak{g} \to \mathfrak{g}$ hyperbolic), these can produce Anosov diffeomorphisms. Such maps are known as *affine diffeomorphisms* of (infra)-nilmanifolds.

Smale's Question (1967)

If a closed smooth manifold M supports an Anosov diffeomorphism $f : M \to M$, must M be homeomorphic to an infranilmanifold?

Previous positive results

Full answer known only in a few special cases:

- Franks and Newhouse (1970, 1970): codimension one Anosov diffeomorphisms only occur on tori.
- Franks and Manning (1969, 1974): Anosov diffeomorphisms on infranilmanifolds are conjugate to affine Anosov diffeomorphisms.
- Brin and Manning (1977, 1981): "sufficiently pinched" Anosov diffeomorphisms can only exist on infranilmanifolds.

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Cautionary examples showing that "homeomorphic" cannot be promoted to "diffeomorphic":

- Farrell and Jones (1978): Anosov diffeomorphisms can exist on certain exotic tori.
- Farrell and Gogolev (2012): Anosov diffeomorphisms on exotic infranilmanifolds.

Previous negative results

Some obstructions to having Anosov diffeomorphisms are also known:

- Shiraiwa (1973): Anosov diffeomorphism with orientable stable (or unstable) distribution cannot induce the identity map on homology in all dimensions.
- Ruelle and Sullivan (1975): if M admits a codimension k transitive Anosov diffeomorphism f with orientable E^u, E^S, then $H^k(M;\mathbb{R}) \neq 0$.
- Yano (1983): if M supports a negatively curved Riemannian metric, then it cannot support an Anosov diffeomorphism.
- Gogolev and Rodriguez Hertz (2014): certain products of spheres cannot support Anosov diffeomorphisms.

Special case of interest

We are primarily interested in analyzing Smale's question in the context of closed aspherical manifolds. For example, here are some specific (currently open) special cases of Smale's Question:

1. M a non-positively curved locally symmetric space,
2. M a product of higher genus surfaces,
3. M a solvmanifold (which is not a nilmanifold).

In each of these cases, Smale's Question predicts that there are no Anosov diffeomorphisms...

Counting periodic points - I

The Lefschetz formula calculates the sum of indices of the fixed points — the Lefschetz number — as follows

$$\Lambda(f) \stackrel{\text{def}}{=} \sum_{p \in Fix(f)} ind_f(p) = \sum_{k \geq 0} (-1)^k \, Tr(f_*|_{H_k(M;\mathbb{R})}).$$

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If M is a closed oriented manifold and f is an Anosov diffeomorphism with oriented unstable subbundle E^u, and f preserves the orientation of the unstable subbundle, then

$$ind_{f^m}(x) = (-1)^{\dim E^u}$$

for all $x \in Fix(f^m)$, $m \geq 1$.

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$$ind_{f^m}(x) = (-1)^{\dim E^u}$$

for all $x \in Fix(f^m)$, $m \geq 1$.This gives us the formula:

$$|Fix(f^m)| = |\Lambda(f^m)|$$

Counting periodic points - II

$|Fix(f^m)|$ can also be calculated from the dynamics. Sinai (1968) and Bowen (1975) constructed Markov codings for Anosov diffeomorphisms. For a transitive Anosov diffeomorphism f the following asymptotic formula holds

$$|Fix(f^m)| = e^{mh_{top}(f)} + o(e^{mh_{top}(f)}),$$

where $h_{top}(f) > 0$ is the topological entropy of f.

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where $h_{top}(f) > 0$ is the topological entropy of f.

Corollary (Shiraiwa's Theorem)

If f is Anosov, it cannot induce the identity map on all homology groups.

Note that this crude approach does not allow us to recover the more refined Ruelle and Sullivan result.

Counting periodic points - Applications

For an aspherical M, we can use group cohomology to compute the Lefschetz number. Since inner automorphisms act trivially on group cohomology, we get:

Corollary

If M is aspherical, and $f : M \to M$ is Anosov, then $Out(\pi_1(M))$ must be infinite.

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Proof.

Out($\pi_1(M)$) is infinite, so $\Gamma := \pi_1(M)$ splits over a 2-ended subgroup [Paulin (1991) and Bestvina-Feighn (1995)], hence $\partial^\infty \Gamma = \partial^\infty \tilde{M} = S^{n-1}$ has a local cutpoint [Bowditch (1998)], a contradiction. □

Need aspherical manifolds M with $\mathrm{Out}(\pi_1(M))$ infinite. Examples of these include higher genus surfaces. Higher dimensional examples include products $M_1 \times M_2$ where one of the M_i has infinite $\mathrm{Out}(\pi_1(M_i))$. Other examples include certain manifolds which split over a torus $M = M_1 \cup_{T^{n-1}} M_2$.

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Question

Let M be an aspherical manifold of dimension ≥ 3, and assume $\mathrm{Out}(\pi_1(M))$ is infinite. Must $\pi_1(M)$ contain a $\mathbb{Z}^2$ subgroup?

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Question

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This is true in dimension $= 3$ (uses Perelman's results on geometrization). If M is assumed to support a locally CAT(0) metric, then Gromov conjectured that no $\mathbb{Z}^2$-subgroup implies $\pi_1(M)$ is Gromov hyperbolic (and then $\mathrm{Out}(\pi_1(M))$ must be finite).

Statement of Main Theorem

Theorem (Gogolev - L.)

Let N be a closed infranilmanifold and let M be a smooth aspherical manifold whose fundamental group $\Gamma = \pi_1(M)$ has the following three properties: (i) Γ is Hopfian, (ii) $Out(\Gamma)$ is finite, and (iii) the intersection of all maximal nilpotent subgroups of Γ is trivial. Then $M \times N$ does not support Anosov diffeomorphisms.

Recall that a group Γ is *Hopfian* if every surjective homomorphism $\phi : \Gamma \to \Gamma$ is automatically an isomorphism.

Sketch of Main Theorem - I

Idea: use the Lefschetz number estimate to get a contradiction. So replace original f by a homotopic map whose Lefschetz numbers we can control. Write $\Gamma := \pi_1(M)$ and $G = \pi_1(N)$, so $\Gamma \times G = \pi_1(M \times N)$.

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Step 1: Analyze map on π_1. From the algebraic hypothesis (iii), $\{1\} \times G$ is a characteristic subgroup, so maps to itself under $f_\#$. Some elementary algebra, along with hypothesis (i), gives you that

$$f_\#((\gamma, g)) = (\alpha(\gamma), \rho(\gamma)L(g))$$

where $\alpha \in \mathrm{Aut}(\Gamma)$, $L \in \mathrm{Aut}(G)$, and $\rho : \Gamma \to \mathcal{Z}(G)$ is a homomorphism into the center of G. By hypothesis (ii), can pass to a power to ensure α is an *inner* automorphism.

Sketch of Main Theorem - II

Step 2: Construct a model map. Using work of Mal'cev (1949), $N = \tilde{N}/G$ where $\tilde{N}$ is a simply connected Nilpotent Lie group, $G \leq \tilde{N}$ a cocompact lattice, and $L : G \to G$ *extends* to an automorphism of N (also denoted L).

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Extend the homomorphism $\rho : \Gamma \to \mathcal{Z}(G)$ to a smooth equivariant map $\rho : \tilde{M} \to \mathcal{Z}(\tilde{N})$ (the center of N is contractible). Then construct a map

$$\tilde{M} \times \tilde{N} \ni (x, y) \mapsto (x, \rho(x)L(y)) \in \tilde{M} \times \tilde{N}.$$

Easy to check that this descends to a model smooth map $\bar{f} : M \times N \to M \times N$, homotopic to f, of the form $\bar{f}(x, y) = (x, \rho(x)L(y))$.

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Key Point:

The map $\bar{f}$ is the identity on the M factor, and only moves elements in the individual N-fibers.

Sketch of Main Theorem - III

Step 3: Perturb the map $\bar{f}$ to get isolated hyperbolic fixed points. Pick a gradient vector field on M with finitely many hyperbolic fixed points $q_1, \ldots, q_k$, perturb the map $\bar{f}$ by moving the fibers along the vector field. Now only fixed points lie above the fibers $\{q_i\} \times N$. Along each fiber, map looks like L. With some care, get the estimate:

$$\Lambda(f^m) = \chi(M)\Lambda(L^m)$$

In particular, if $\chi(M) = 0$ we already have a contradiction!

Sketch of Main Theorem - III

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Unfortunately, if $\chi(M) \neq 0$, then there is a lot more work needed. In this case, the formula above can be used to verify that L is actually an Anosov automorphism of N (this uses work of Manning (1974)).

Sketch of Main Theorem - IV

Step 4: Lifting and compactifying. In the case where $\chi(M) \neq 0$, one performs a further coordinate change, and use some dynamics results (Frank's thesis (1970), basic properties of hyperbolicity) to ensure that the perturbed map has a locally maximal hyperbolic set K with infinitely many periodic points.

Sketch of Main Theorem - IV

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Then lift this new map to a map $\tilde{f} : M \times \tilde{N} \to M \times \tilde{N}$, and verify that there is a lift $\tilde{K}$ of K which is $\tilde{f}$-invariant and still exhibits hyperbolic behavior. Away from this set, homotope $\tilde{f}$ to coincide with $L : \tilde{N} \to \tilde{N}$ on each fiber. Finally, compactify each $\tilde{N}$-fiber with one point at infinity, obtaining a map $\hat{f} : M \times S^k \to M \times S^k$.

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Key Point:

The number of periodic points in $\tilde{K}$ still grows exponentially.

Sketch of Main Theorem - V

Step 5: Counting periodic points of $\hat{f}$. Finally, we perturb the map $\hat{f}$ near the fibers "at infinity" $M \times \{\infty\}$ in order to get finitely many additional hyperbolic fixed points near infinity. Then we can calculate the Lefschetz numbers of $\hat{f}$ via:

$$\Lambda(\hat{f}^m) = \sum_{p \in Fix(\hat{f}^m)} ind_{\hat{f}^m}(p) = (-1)^u \chi(M) + (-1)^{\dim E^s} \left| Fix(\hat{f}^m|_{\tilde{K}}) \right|,$$

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an expression which grows exponentially.

On the other hand, it is not too hard to check that $\hat{f}$ is homotopic to the identity map – so we also have that $\Lambda(\hat{f}^m) = \chi(M)\chi(S^k)$, a constant. This contradiction completes the sketch of the proof.

Main Example

Corollary (Gogolev - L.)

Let N be any closed infranilmanifold, and let $M_1, \ldots, M_k$ be a collection of closed smooth aspherical manifolds of dimension ≥ 3, each of which satisfies one of the following properties:

1. *it has Gromov hyperbolic fundamental group; or*
2. *it is an irreducible higher rank locally symmetric space with no local $\mathbb{H}^2$-factors or $\mathbb{R}$-factors.*

Then the product $M_1 \times \cdots M_k \times N$ does not support any Anosov diffeomorphisms.

Note that these are the simplest example which are *not* covered by the easy arguments mentioned earlier – these can have infinite outer automorphism groups (due to the N factor).

Sketch of Corollary - I

For the corollary, we just need to check that $\pi_1(M_1 \times \cdots \times M_k)$ satisfies the three algebraic conditions:

1. Γ is Hopfian,
2. $Out(\Gamma)$ is finite, and
3. the intersection of all maximal nilpotent subgroups of Γ is trivial.

Case 1: Single hyperbolic factor. For Γ a hyperbolic group, (1) follows from Sela (1999). (2) follows by a similar argument as our proof of Yano's theorem (using Paulin, Bestvina-Feighn, and Bowditch); in this case $\partial^\infty\Gamma$ might not be a sphere, but one can still rule out local cutpoints. And (3) follows since the only nilpotent subgroups of Γ are the cyclic ones.

Sketch of Corollary - II

Case 2: Single locally symmetric factor. For Γ a lattice in a higher rank semi-simple Lie group, (1) follows from linearity and work of Mal'cev (1940). (2) follows from Mostow's rigidity theorem. And (3) is an immediate consequence of Margulis' normal subgroup theorem.

Case 3: Multiple factors. Property (iii) can be easily deduced from the corresponding property for the individual factors. Property (i) and (ii) require some work. Key to both of these is to analyze surjective homomorphisms of the product $\Gamma_1 \times \cdots \times \Gamma_k$ to itself. One shows that such a map is given by surjective homomorphisms from individual Γ_i onto some other $\Gamma_{\tau(i)}$, where $\tau \in \mathrm{Sym}(k)$. Property (i) and (ii) easily follow.

Thank You!