Quasimorphisms on diffeomorphism groups

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Outline

1. Main theme of the talk
2. Norms/metrics on $\mathrm{Diff}_0(M)$ e.g. commutator length norm, fragmentation norm
3. Known results for various manifolds M
4. Our results
5. Quasimorphisms on $\mathrm{Diff}_0(S)$
6. Remarks and overview of the proof

Tea break, followed by more detailed discussions.

Notation

- M smooth manifold
- $\mathrm{Diff}(M)$ orientation-preserving compactly-supported diffeomorphisms $M \to M$, a topological group
- $\mathrm{Diff}_0(M)$ the component of Diff containing the identity
- Today: Interested in the algebra of $\mathrm{Diff}_0(S)$, surface S
- Results/techniques of talk also apply to $\mathrm{Homeo}_0(S)$.

Main theme of the talk

S an orientable closed surface genus ≥ 1. There is an exact sequence

$$\mathrm{Diff}_0(S) \to \mathrm{Diff}(S) \to \mathrm{Mcg}(S)$$

$\mathrm{Mcg}(S)$ is countable, has uncountably many normal subgroups well understood

$\mathrm{Diff}_0(S)$ is uncountable, has only two normal subgroups poorly understood

Despite these key differences we introduce $\mathrm{Mcg}(S)$-inspired tools for $\mathrm{Diff}_0(S)$.

Background: fragmentation norm

$f \in \mathrm{Diff}_0(M)$ define

$$\mathrm{supp}(f) := \overline{\{x \in M : fx \neq x\}}$$

Say f is *disk supported* if $\mathrm{supp}(f) \subset B$ open disk B.

Lemma (Fragmentation Lemma)

$\forall f \in \mathrm{Diff}_0(M)\ \exists f_1, \dots, f_n \in \mathrm{Diff}_0(M)$ such that $f = f_1 \dots f_n$ and each f_i disk supported.

i.e. the disk-supported maps generate $\mathrm{Diff}_0(M)$

Definition (Fragmentation norm frag)

$\mathrm{frag}(f)$ is the word length of f with the disk-supported maps as the generating set i.e.

$$\mathrm{frag}(f) := \min\{n : f_1, \dots, f_n \text{ as above}\}$$

and $\mathrm{frag}(id) := 0$.

Fragmentation norm isn't easy to understand

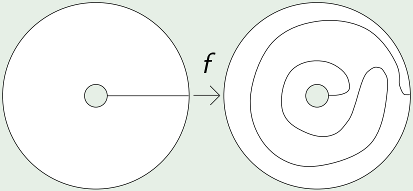
- $\text{frag}(f) = 0$ iff $f = id$
- $\text{frag}(f) = 1$ iff $f \neq id$ and f disk supported
- $\text{frag}(f) \geq 2$ iff f not disk supported

What more can we say ...?

In general computing the fragmentation norm is poorly understood.

Example

$M = S^1 \times (-1, 1)$ point push around the core curve $S^1 \times \{0\}$.



How does $\text{frag}(f^n)$ with respect to n? It's at most linear, but is it linear? Sublinear? Bounded?

Another norm: the commutator length

Mather, Thurston: $\mathrm{Diff}_0(M)$ is perfect i.e. every element f is a *product* of commutators $[a,b] = aba^{-1}b^{-1}$. Equivalently any homomorphism of $\mathrm{Diff}_0(M)$ to an abelian group has trivial image.

$\mathrm{Homeo}_0(M)$ is also perfect. Both groups are simple too!

Thus we can also consider

Definition (Commutator length norm cl)

$\mathrm{cl}(f)$ is the word length of f with respect to the generating set the set of commutators, and $\mathrm{cl}(id) := 0$

Natural Question

When is $\mathrm{cl}\colon \mathrm{Diff}_0(M) \to \mathbb{R}$ *a bounded function*
i.e. when is $\mathrm{Diff}_0(M)$ *uniformly perfect?*

Work of Burago–Ivanov–Polterovich

frag and cl are examples of *conjugation-invariant norms* on $\mathrm{Diff}_0(M)$

For many M these norms are uniformly bounded.

Theorem (Burago–Ivanov–Polterovich)

Let M be one of the following

- *an open n-ball B,*
- *an open annulus $S^1 \times (-1, 1)$,*
- *more generally a portable manifold e.g. open handlebody,*
- *an n-sphere S^n, or*
- *a closed, orientable 3-manifold.*

Then for any conjugation-invariant norm $\rho\colon \mathrm{Diff}_0(M) \to \mathbb{R}$ there exists C such that $\forall f \in \mathrm{Diff}_0(M)$ we have $\rho(f) \leq C$.
Thus frag *is bounded and* $\mathrm{Diff}_0(M)$ *is uniformly perfect.*

So frag on the annulus is uniformly bounded! *Not* obvious!

frag is natural in an algebraic sense

We obtain

Corollary (Burago–Ivanov–Polterovich)

Given a conjugation-invariant norm $\rho\colon \mathrm{Diff}_0(M) \to \mathbb{R}$ *there exists* C *such that* $\forall f$ *we have* $\rho(f) \leq C\,\mathrm{frag}(f)$*.*

The proof is beautifully short. Write $f = f_1 \dots f_n$ where $n = \mathrm{frag}(f)$ and f_i are disk supported. Then by the previous theorem $\rho(f_i) \leq C$ for some C (hint: after conjugating f_i they are supported in the same ball, use conjugation invariance), so we're done.

frag despite its purely topological definition is in fact intimately related to the algebra of $\mathrm{Diff}_0(M)$

Natural Question

But is there M *with* frag *unbounded on* $\mathrm{Diff}_0(M)$*?*

More bounded examples

Theorem (Tsuboi)

Let M be an orientable closed smooth n-manifold with either

- $n \geq 5$, *or*
- $n = 4$ *such that M has a handlebody decomposition without 2-handles,*

then $\mathrm{Diff}_0(M)$ *is uniformly perfect and* frag *is bounded.*

We are left with dimensions 2 and 4:

Question

What about closed surfaces S with positive genus?
What about 4-dimensional M with 2-handles?

Main theorem

Let G be a perfect group. The *stable commutator length* of $g \in G$ is $\mathrm{scl}(g) := \lim_{n\to\infty} \frac{1}{n}\,\mathrm{cl}(g^n)$.

Theorem (Bowden–Hensel–W.)

Let S be a closed, orientable surface of genus ≥ 1. Then there exists $g \in \mathrm{Diff}_0(S)$ with $\mathrm{scl}(g) > 0$. In particular $\mathrm{Diff}_0(S)$ is not uniformly perfect and frag *is unbounded.*

1. These are the first examples of closed smooth manifolds M where $\mathrm{Diff}_0(M)$ has unbounded fragmentation norm.
2. Our methods/results apply to $\mathrm{Homeo}_0(S)$ too.

Strategy: Construct *quasimorphisms* to show $\mathrm{scl}(g) > 0$ and cl unbounded and deduce frag unbounded.

These quasimorphisms come from *geometric group theory*.

Quasimorphisms

Definition

Let G be a group. A map $\phi\colon G \to \mathbb{R}$ is a *quasimorphism* if $\exists D \geq 0$ such that $\forall g, h \in G$ we have

$$|\phi(gh) - \phi(g) - \phi(h)| \leq D.$$

This is a tool to show scl > 0 somewhere and hence cl unbounded. Observe:

1. $|\phi(id)| \leq D$,
2. $|\phi[f, g]| \leq 7D$, so commutators are bounded,
3. if ϕ is unbounded then $\exists g \in G$ with $\phi(g^n)$ linear in n,
4. such g has $\mathrm{scl}(g) > 0$ hence G not uniformly perfect.

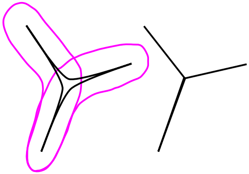
There is a converse called *Bavard duality*.

Source of quasimorphisms via hyperbolic geometry

To prove our theorem we need an unbounded quasimorphism. This will come from an action of $\mathrm{Diff}_0(S)$ by isometries on a hyperbolic space $\mathcal{C}^\dagger(S)$, defined later.

Definition (Hyperbolicity)

A geodesic metric space X is *hyperbolic* if $\exists \delta \geq 0$ such that any triangle formed of geodesics g_1, g_2, g_3 satisfies $g_1 \subset N_\delta(g_2 \cup g_3)$.



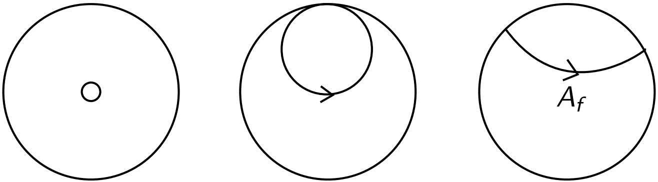
Plays an important role in some breakthroughs of the past decade
Cremona group is not simple (Cantat–Lamy)
Virtual fibring/Haken conjecture in 3-manifolds (Agol, Wise)

Classification of isometries (think $\mathbb{H}^2$)

Define $|f| := \lim_{n\to\infty} \frac{1}{n} d(x, f^n x)$ *asymptotic translation length*.
Any isometry of a hyperbolic space is either

- *elliptic*, $|f| = 0$ and has bounded diameter orbit, or
- *parabolic*, $|f| = 0$ has unbounded diameter orbit, or
- *loxodromic*, $|f| > 0$.

$\mathrm{Diff}_0(S)$ acts by isometries on $\mathcal{C}^\dagger(S)$.



Today: Loxodromic elements are the important ones.

Bestvina–Fujiwara condition for quasimorphisms

Here is a widely applicable and useful condition to produce many unbounded quasimorphisms, which generalises that of Epstein–Fujiwara and Brooks.

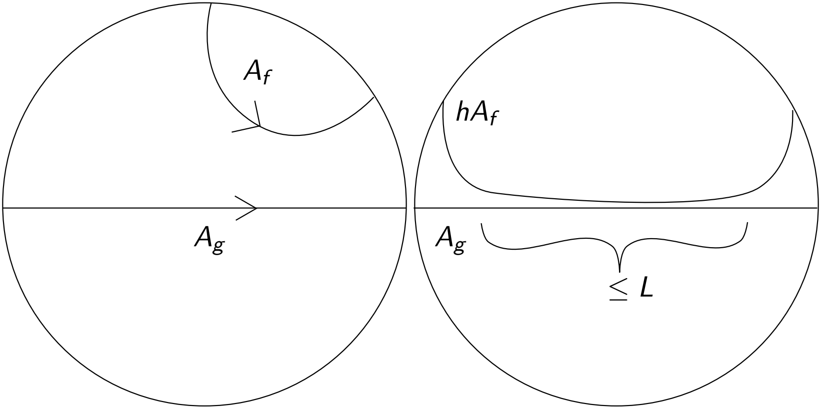
Theorem (Bestvina–Fujiwara, (BF) condition)

Suppose G act by isometries on hyperbolic X with loxodromic elements $f, g \in G$ that are independent. Then there exists many unbounded quasimorphisms on G.

Technical: Here we say that loxodromic f and g are *independent* if there is a uniform bound L such that for any $h \in G$, hA_f cannot fellow travel A_g for further than L, where A_f and A_g are the "axes" of f and g.

Non-example: natural action of $PSL_2\mathbb{R}$ on $\mathbb{H}^2$.
Example: natural action of $PSL_2\mathbb{Z}$ on $\mathbb{H}^2$.

(BF) condition

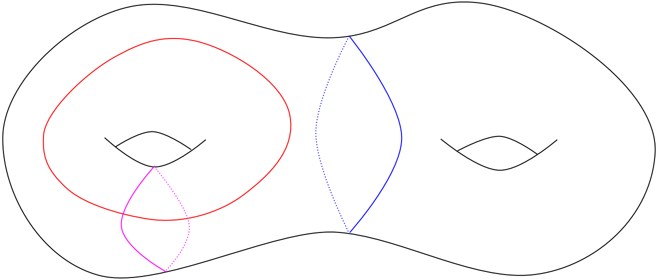


The hyperbolic space $\mathcal{C}^\dagger(S)$

Definition (Diffeomorphism curve graph of S)

For genus ≥ 2 we define $\mathcal{C}^\dagger(S)$ to be the graph with
Vertices: smooth simple closed curves in S, not null-homotopic
Edges: Between $\alpha \neq \beta$ iff $\alpha \cap \beta = \emptyset$.

Allow slightly more edges when genus $= 1$.
This is a metric space when we set each edge length equal to 1.



Main theorem

The main theorem now follows from

Theorem (Bowden–Hensel–W.)

$\mathcal{C}^\dagger(S)$ *is hyperbolic and the natural action of* $\mathrm{Diff}_0(S)$ *on* $\mathcal{C}^\dagger(S)$ *satisfies* (BF)

Thus cl and therefore frag are unbounded.

Is this the first example of a simple group with action on a hyperbolic space satisfying (BF)?

Overview of the proof

Hyperbolicity of $\mathcal{C}^\dagger(S)$:
We show that distances between (transverse) vertices/curves are realised as distances in δ-hyperbolic spaces for absolute δ. (these spaces are types of ordinary curve graphs) This implies hyperbolicity of $\mathcal{C}^\dagger(S)$.

Loxodromics:
Examples come from point-pushing pseudo-Anosovs on $S - P$, where $P \subset S$ is finite. The distance realisation from above helps prove this.

Independent loxodromics:
Let $f \in \mathrm{Diff}_0(S)$ be loxodromic on $\mathcal{C}^\dagger(S)$. Fix representative $b \in \mathrm{Diff}(S)$ of a pseudo-Anosov mapping class. Then we show that for $n > 0$ sufficiently large we have f and $g = b^n f b^{-n}$ independent in $\mathrm{Diff}_0(S)$.

Tea break

$\mathcal{C}^\dagger(S)$ hyperbolic

We're going to compare distances in $\mathcal{C}^\dagger(S)$ to distances in $\mathcal{C}^s(S-P)$, $P \subset S$ finite.

Definition (Surviving curve graph)

Let S be a closed surface genus ≥ 2 and $P \subset S$ finite. The vertices of $\mathcal{C}^s(S-P)$ are the isotopy classes of *surviving* simple closed curves in $S-P$ i.e. curves that are not null-homotopic in S. Edges if they admit disjoint representatives.

This is a subgraph of the curve graph $\mathcal{C}(S-P)$, which was proved to be hyperbolic by Masur–Minsky. Curve graphs were proved to be hyperbolic with an absolute constant δ (Aougab, Bowditch, Clay–Rafi–Schleimer, Hensel–Przytycki–W., Przytycki–Sisto).

Theorem (A. Rasmussen)

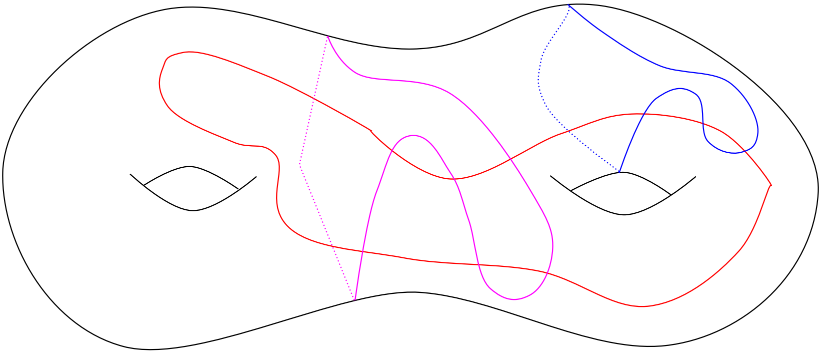
There is δ such that $\mathcal{C}^s(S-P)$ is δ-hyperbolic whenever S closed genus ≥ 2 and $P \subset S$ finite.

$\mathcal{C}^\dagger(S)$ hyperbolic

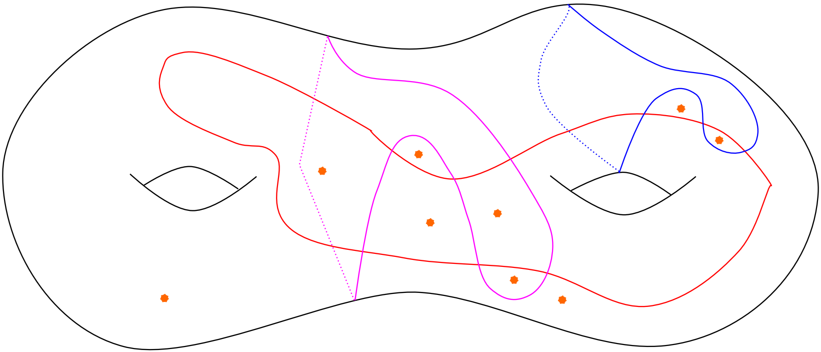
Four-point condition: if there is δ such that for any set of points $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \subset \mathcal{C}^\dagger(S)$ we have that $d^\dagger(\alpha_i, \alpha_j)$ is uniformly approximated by $d(v_i, v_j)$ where v_i are in some δ-hyperbolic space then $\mathcal{C}^\dagger(S)$ is hyperbolic.
Overview:

1. Pay a small price (errors of ± 2) to make the α_i pairwise transverse.
2. Puncture each component of $S - \cup_i \alpha_i$. Puncture set $P \subset S$ finite.
3. Then $d^\dagger(\alpha_i, \alpha_j) = d_{\mathcal{C}^s(S-P)}([\alpha_i]_{S-P}, [\alpha_j]_{S-P})$.
4. We're done by A. Rasmussen's theorem and the four-point condition.

Puncturing the complementary regions



Puncturing the complementary regions



puncture set P in orange

$\mathcal{C}^\dagger(S)$ hyperbolic: $d^\dagger(\alpha_i, \alpha_j) = d_{\mathcal{C}^s(S-P)}([\alpha_i]_{S-P}, [\alpha_j]_{S-P})$

We can project α_i to its isotopy class $[\alpha_i]_{S-P}$. Disjointness is preserved thus

$$d^\dagger(\alpha_i, \alpha_j) \geq d_{\mathcal{C}^s(S-P)}([\alpha_i]_{S-P}, [\alpha_j]_{S-P}).$$

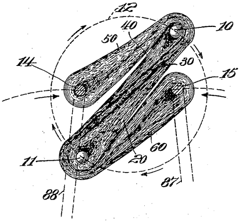
On the other hand, if α_i and α_j are in *minimal position* on $S - P$, then given a geodesic $[\alpha_i]_{S-P}, v_1, \ldots, v_{d-1}, [\alpha_j]_{S-P}$ in $\mathcal{C}^s(S-P)$, we can find representatives $\alpha_i, \nu_1, \ldots, \nu_{d-1}, \alpha_j$ which gives a path in $\mathcal{C}^\dagger(S)$. Hence

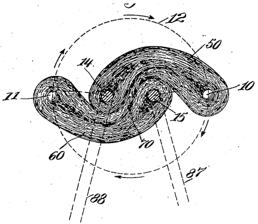
$$d^\dagger(\alpha_i, \alpha_j) \leq d_{\mathcal{C}^s(S-P)}([\alpha_i]_{S-P}, [\alpha_j]_{S-P}).$$

(puncturing each complementary region of $S - \cup_i \alpha_i$ guarantees minimal position)

Loxodromics on $\mathcal{C}^\dagger(S)$ in $\mathrm{Diff}_0(S)$

A mapping class of $S - P$ is *pseudo-Anosov* if it has a representative which is Anosov outside finitely many points.





Theorem (Masur–Minsky)

$f \in \mathrm{Mcg}(S - P)$ *pseudo-Anosov*
$\implies$ *f loxodromic on $\mathcal{C}(S - P)$*

$\implies$ f loxodromic on $\mathcal{C}^s(S - P)$ too.
By the Lipschitz projection $\mathcal{C}^\dagger(S)$ to $\mathcal{C}^s(S - P)$ we get that *any* representative of f as a diffeomorphism of S preserving P will be loxodromic on $\mathcal{C}^\dagger(S)$

Independent loxodromics

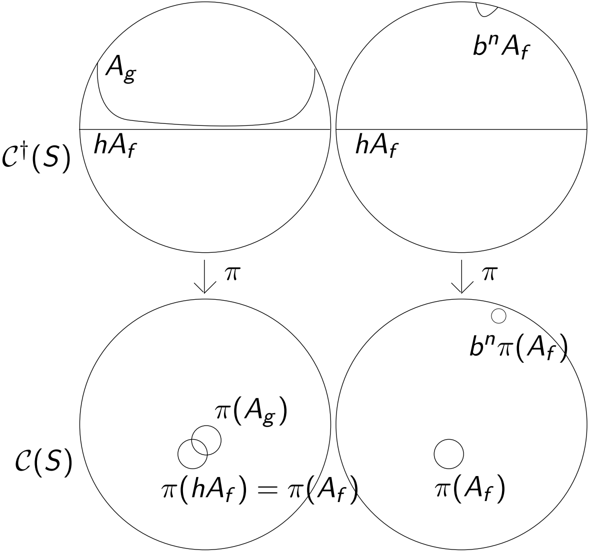
Observe there is a 1-Lipschitz map $\pi\colon \mathcal{C}^\dagger(S) \to \mathcal{C}(S)$, which is Diff$(S)$-equivariant.
Furthermore for $f \in \mathrm{Diff}_0(S)$ then $f[\alpha] = [\alpha]$ for every $[\alpha]$ in $\mathcal{C}(S)$ i.e. $\mathrm{Diff}_0(S)$ does nothing to $\mathcal{C}(S)$.

Lemma

If f and g are dependent loxodromic elements of $\mathrm{Diff}_0(S)$ then we can find $h \in \mathrm{Diff}_0(S)$ such that hA_f fellow travels A_g for a long time in $\mathcal{C}^\dagger(S)$.
Therefore $\pi(hA_f) = h\pi(A_f) = \pi(A_f)$ and $\pi(A_g)$ are uniformly bounded from each other in $\mathcal{C}(S)$. (constants depending on δ and quality of A_f and A_g)

However for $b \in \mathrm{Diff}(S)$ we have $A_{b^n f b^{-n}} = bA_f$ and $\pi(bA_f) = b\pi(A_f)$ can be made arbitrarily far from $\pi(A_f)$.
Therefore if we pick b pseudo-Anosov and n large enough we will have f and $g = b^n f b^{-n}$ independent in $\mathrm{Diff}_0(S)$.

Cartoon of independent loxodromics



Thank you!