A surface embedding theorem

joint with Daniel Kasprzycki
Mark Powell
Peter Teichner
Q: Given a map of a surface in a 4-\text{mfld}, when is it homotopic to a (loc.-flat) embedding?

\[ (u, u \cap \Sigma) \cong (\mathbb{R}^4, \mathbb{R}^2) \]

\[ \text{homeo} \]

\[ \Sigma \]

e.g., which elements of \( \pi_2(M^4) \) are rep. by embedded spheres?

- When does a knot in \( S^3 \) bound an embedded disc in \( B^4 \)?
Prototypical result: Disc embedding theorem

Let $M^4$ be a connected topological manifold, $\pi_1 M$ good.

$\Sigma = \cup \Sigma_i$: compact surface, each $\Sigma_i$ simply connected

$$F: \Sigma \hookrightarrow M$$

is a generic immersion such that

- the algebraic intersection numbers of $F$ vanish
- $F: \Sigma \hookrightarrow M$ framed, alg. dual spheres for $F$

Then $F$ is (neg.) htpic rel $\partial$ to a loc. flat embedding $\overline{F}$

[with geom dual spheres $\overline{G}$ s.t. $\overline{G} \simeq G.$] $\pi_1 \neq 1$

Powell-R-Teichner'20.
Generic immersions (loс. flāт):

- locally an embedding, intersections isolated double points \((2 + 2 = 4)\)

- any continuous map \(\Sigma^2 \rightarrow M^4\) is htpic to a gen. imm.
  
  \[\text{[FQ; see PRT'}^{20}\text{]}\]
  
  uses that any noncompact, connected 4-mfld is smoothable.

[Quinn]

Good groups

- abelian gps, finite gps, solvable gps, ...

- gps of snbexp growth

- closed under subgps, quotients, extensions, direct limits

- open whether all gps are good e.g. \(\mathbb{Z} \ast \mathbb{Z}\)?
Intersection numbers

\[ \lambda(f,g) := \sum_{p \in f \cap g} \varepsilon(p) Y(p) \quad \in \mathbb{Z} [\pi_1 M] \]

well defined if \( f, g \) are rimp. connected (modulo whiskers)

\[ \lambda(f,g) = 0 \iff \text{all points in } f \cap g \text{ are paired by gen. immersed discs, with (framed) disjoint, embedded boundaries} \]

Self-intersection number \( \mu(f) = 0 \iff \text{all pts in } f \cap f \text{ are paired by gen. coll. of 0-discs} \)

\( f, g \) are alg dual if \( \lambda(f,g) = 1 \iff \text{all but one pt in } f \cap g \text{ are so paired} \)

\( f, g \) are geom dual if \( f \cap g = 1 \text{pt} \)
The Whitney trick

$t = -2$

$t = 0$

$t = 3$

$\mathbb{R}^4 \cong \mathbb{R}^3 \times \text{time } t$
The Whitney trick

\[ \text{A} \]

\[ t = 0 \]

\[ t = -2 \]

\[ t = 3 \]
Disc embedding theorem

Casson, Freedman ’82, Freedman-Quinn ’90
Stong ’94, Kasprowski-Powell-R-Teichner

$\Sigma = \bigcup \Sigma_i$ compact surface, each $\Sigma_i$ simply connected

$E = \bigcup E_i$ compact surface, each $E_i$ $\pi_1 M$-good.

$F: \Sigma \rightarrow M$

$\Sigma \hookrightarrow \Sigma \rightarrow \partial M$

a generic immersion

such that • algebraic intersection numbers of $F$ vanish

$E: \Sigma^2 \rightarrow M$ framed, alg. dual spheres for $F$

Then $F$ is neg. htpic rel $\partial$ to a loc. flat embedding $F$

with geom dual spheres $G^*$ s.t. $G \simeq G$.

if and only if the Kervaire-Milnor invariant

$km(F) \neq 7/2 \text{ vanishes}$
RHT is not slice i.e. A emb disc bounded by K.

Every $K \subseteq S^3$ bounds an emb. disc in $\# \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$

given $K$, min $m$ s.t. $K$ null-hom slice in $m \mathbb{CP}^2$.

null-hom disc in $\# S^2 \times S^2$ iff $\operatorname{Arf}(K) = 0$.
Corollary 1: $F: \Sigma^2 \to M^4$ with $\pi_1 M$ good.

$F'$ is the result of adding a trivial handle to $F$. Then $F'$ is (reg) htpic to an embedding.

Corollary 2: $F: \Sigma^2 \to M^4$ with $\sim S^2$.

$F$ is (reg) htpic to an embedding.

Corollary [FMNOPR]: $g_{2h}\left(\kappa\right) \leq 1$. 
Intersection numbers

$\lambda(f,g)$ not well defined in $\mathbb{Z}[\pi_1 M]$!
but count in a double coset space

$\lambda(f,g) = 0 \iff$ all pts in $f \cup g$
paired by gen imm
coll of Wh discs

$M(f) = 0 \iff$ all pts in $f \cup f$
paired by gen imm
coll of Wh discs
The Kervaire–Milnor invariant
[for discs/spheres, due to FQ90 §10+strong]

\[ \Sigma = \bigcup \Sigma_i \]

\[ F : \Sigma \to M \text{ trivial alg int numbers, } \exists G : L \mathbb{S}^2 \to M \text{ alg dual} \]

\[ \Rightarrow \text{ if } f \text{ are paired by gen. coll. of Wh discs } W \]

Let \( \Sigma^0 \subseteq \Sigma \) subsurface, \( F^0 = F|_{\Sigma^0} \) admits only twisted duals

i.e. euler number of the norm. bundles are odd.

Let \( W^0 = \{ W^{0,0} \} \subseteq W \) subset pairing into of \( F^{0} \).

Then \( \text{Rm}(F, W) := \sum_{Q} \left| \text{Int} W^{0,0} \cap F^{0} \right| \text{ mod } 2 \).
Question: When is $km(F, W)$ independent of $W$?

(spoiler: when $F$ is $b$-characteristic)
Proof outline: Suppose \( \exists W \) s.t. \( \text{Rm}(F, W) = 0 \text{\&} \frac{76}{12} \)

**Step 1:** By neg. htpy, make \( F \) and \( G \) \textit{geom dual} (still immersed) \((\text{standard trick})\)

**Step 2:** Upgrade \( W \) and \( F \) by neg. htpy s.t. \( \{ \text{Int} + \text{We}\} \cap F = \emptyset \)

**Step 3:** Use (Whitney) disc embedding theorem to replace \( W \) by \( \{ \text{Ve}\} \) s.t. \( \{ \text{Int} + \text{Ve}\} \cap F = \emptyset \)
- \( \{ \text{Ve}\} \) flat, embedded, disjoint
- \( \exists \text{geom dual spheres} \{ \text{Ve}\} \) in \( M \setminus F \)

**Step 4:** Tube \( G \) into \( \{ \text{Ve}\} \) to get \( \overline{G} \), geom dual to \( F \), disjoint from \( \{ \text{Ve}\} \)

**Step 5:** Whitney move \( F \) over \( \{ \text{Ve}\} \) to get desired \( \overline{F} \).
Step 2: Upgrade \( W \) and \( F \) by neghtpy s.t. \( \{ \text{Int} \ W \} \cap F = \emptyset \)

- Local cusp moves in \( \text{Int} \ W \) changes framing by \( \pm 2 \).
Step 2: Upgrade \( W \) and \( F \) by neghtpy s.t. \( \{ \text{Int} + W e \} \cap F = \emptyset \)

Remaining problem: \( \text{W} \)h \( d\)is\( c\)s for \( F \) with a single "problem" each

\[ R_m = 0 \implies \text{there are even such problem discs.} \]

- Do a finger move between \( f_2 \) and \( f_5 \)
Thanks!
The diagram shows a plumbing of $D^2 \times D^2$. It is marked as a smooth 4-manifold.

\[ \gamma \text{ is good if } \]

\[ \forall \psi \in \mathcal{G}(1.5)^g \to \Gamma \exists \text{ disc } \Delta \text{ s.t. } \psi(\text{disc}) = e \]

\[ \text{null disc property} \]