TOPOLOGICAL CYCLIC HOMOLOGY FOLLOWING NIKOLAUS–SCHOLZE (WS17/18, [DATE], [ROOM])

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Seminar Description

This is the seminar program for the HIOB 6 seminar at the Univerisity of Regensburg in the winter term 2017/2018. In this seminar we want to learn the new construction of topological cyclic homology due to Thomas Nikolaus and Peter Scholze. We will follow their preprint [1] quite closely. If you are interested in giving a talk, please contact me at

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The main goal will be to understand this new construction, which now does not depend on point set models for THH anymore, and discuss some of the applications proposed in [1, Chapter 4]. We will not discuss the comparison of the new TC and the old TC and will also not discuss the equivalence of the newly constructed ∞-category of cyclotomic spectra to the previously existing one.

The language of ∞-categories will be used heavily. It will be the speakers responsibility to explain what results from ∞-categories are used, and to decide which of those results deserve extra treatment in the talk. Although this is not intended as a seminar on higher categories, there are some talks that intrinsically involve a lot of higher categories (or are about statements in higher categories for that matter), I will indicate them with a "(HC)". The talks that require more background (or the wish to acquire more knowledge) in (equivariant) stable homotopy theory will be indicated with "(SH)".

Seminar Schedule

1. The Tate construction and cyclotomic spectra

Talk 1 (2017 – N.N.): *The Tate construction.* Introduce Tate cohomology of finite groups and give an informal definition of the Tate construction on a spectrum with G action – In particular introduce the ∞-category of spectra with G-action. Give examples of such objects (EM spectra, complex K-theory, Atiyah's KR theory, equivariant bordism, algebraic K-theory, ...) Then explain the construction of the Norm map in general [1, Definition I.1.10]. Explain the Tate spectral sequence and discuss examples such as $\mathrm{H}\mathbb{Z}^{tC_2}$, maybe ku^{tC_2}, and structural results as [1, Lemmas I.2.7 and I.2.8].

Main reference: [1, Chapter I.1 pp. 9–13.]

Talk 2 – **(HC)** (2017 – N.N.): *Multiplicativity of the Tate construction I.* Explain what the goal of the next 2 talks is ([1, Theorem I.3.1]) and say that we will see yet another approach in Talk 4. The rest of the talk will be an elaboration on multiplicative properties of Verdier quotients of stable ∞-categories. The goal is to prove [1, Theorem I.3.6].

Main reference: [1, Chapter I.3 pp. 18–23.]

Talk 3 (2017 – N.N.): *Multiplicativity of the Tate construction II.* Apply the results of talk 2 to the case of interest. Recall the statement of [1, Theorem I.3.1], and then start with [1, Definition I.3.7]. Then explain the proof of [1, Theorem I.3.1] and [1, Corollary I.3.9].

Main reference: [1, Chapter I.3 pp. 23–26.]

Talk 4 – **(SH)** (2017 – N.N.): *Genuine equivariant homotopy theory.* Focus on structural statements: compare G-objects in spectra to functors on the orbit category, and this in turn to genuine G-equivariant spectra (i.e. functors on the effective Burnside category of G), see [1, Remark II.2.6]. Try to highlight what additional structure is available in the 3 different categories. The main points to get at are [1, Proposition II.2.4], [1, Theorem II.2.7] (plus the multiplicativity of that right adjoint), and [1, Proposition II.2.13].

Main reference: [1, Chapter II.2]

Talk 5 (2017 – N.N.): *(p)–Cyclotomic spectra.* Define cyclotomic and p-cyclotomic spectra and discuss briefly what a cyclotomic spectrum was previously. Indicate why a cyclotomic spectrum in the old definition gives a cyclotomic spectrum in the new definition building on talk 4. Then continue to discuss the ∞-category of cyclotomic and p-cyclotomic spectra in terms of lax equalizers. Finish with [1, Definition II.1.8] – the definition of TC of a cyclotomic spectrum, and [1, Proposition II.1.9] – a formula for TC of a cyclotomic spectrum.

Main reference: [1, Chapter II.1 pp. 30–35.]

2. Topological Hochschild Homology

Talk 6 (2017 – N.N.): *The Tate diagonal.* Construct the functor T_p and explain how to obtain the Tate diagonal $X \to (X \otimes \cdots \otimes X)^{tC_p}$ of [1, Definition III.1.4]. Prove [1, Theorem III.1.7] and [1, Theorem III.1.10] which says that the Tate diagonal does not exists in chain complexes (as opposed to spectra).

Main reference: [1, Chapter III.1 pp. 66–71.]

Talk 7 – **(HC)** (2017 – N.N.): *The cyclotomic structure on* THH. The goal of this talk is to construct, for an $\mathbb{E}_1$-ring spectrum A, the topological Hochschild homology $\mathrm{THH}(A)$ with its cyclotomic structure. This is done by means of geometric realization of spectral cyclic objects ([1, Appendix B]). Explain carefully how the cyclotomic structure maps arise (at some point a Tate construction will have to be commuted with geometric realization) and why we need strong functorial properties of the Tate diagonal, i.e. make precise the cyclotomic structure maps modulo the construction of the transformation $I \to \tilde{T}_p$; this will be done in talk 9. It will be worthwhile to talk to the person giving talk 9, as the material to be covered here is more than in talk 9, maybe there is a good way of deferring parts to that talk.

Main reference: [1, Chapter III.2 pp. 72-75 and Appendix B.]

Talk 8 – **(HC)** (2017 – N.N.): *The multiplicativity of the Tate diagonal.* The goal of this talk is to prove a universal property of the identity functor on spectra: It is the initial left exact and lax symmetric monoidal endofunctor of spectra, see [2, Corollary 6.9]. Give an overview of the ideas of this paper and how they are used to prove the result we need.

Main reference: [2, Section 6]

Talk 9 – **(HC)** (2017 – N.N.): *The cyclotomic structure of* THH *– the technical lemma.* The main body of this talk is of technical nature: Construct the transformation $I \to \tilde{T}_p$ which was used in talk 7 to construct the cyclotomic structure maps on THH. Main reference: [1, Chapter III.3 pp. 75–80.]

3. Examples

Talk 10 – **(SH)** (2017 – N.N.): *The Tate-valued Frobenius I.* Define the Tate-valued Frobenius for $\mathbb{E}_\infty$-rings. Then start to relate the Tate-valued Frobenius to power operations and go through the example of complex K-theory, [1, Proposition IV.1.12] and end with the statement of [1, Theorem IV.1.15].

Main reference: [1, Chapter IV.1 pp. 96–102.]

Talk 11 – **(SH)** (2018 – N.N.): *The Tate-valued Frobenius II.* The goal of this talk is to prove [1, Theorem IV.1.15], i.e. to describe the Tate-valued Frobenius for $\mathrm{H}\mathbb{F}_p$ in terms of Steenrod (Power) operations. This requires the whole rest of the section.

Main reference: [1, Chapter IV.1 pp. 103–109.]

Talk 12 (2018 – N.N.): THH *of* $\mathbb{E}_\infty$*-algebras.* Recall the Tate-valued Frobenius of $\mathbb{E}_\infty$-rings. Explain why its existence does not imply that every $\mathbb{E}_\infty$-ring acquires a canonical cyclotomic structure (think about [1, Corollary IV.2.4]). Then continue to explain why this turns out to work for THH(A) if A is $\mathbb{E}_\infty$, i.e. explain [1, Corollary IV.2.3].

Main reference: [1, Chapter IV.2 pp. 109–112.]

Talk 13 (2018 – N.N.): THH *and* TC *of spherical group rings.* The goal is to calculate TC and THH of spherical groups rings $\mathbb{S}[\Omega Y]$. For this, first work towards THH($\mathbb{S}[\Omega Y]$) by discussing the cyclic Bar construction in (pointed) spaces, [1, Proposition IV.3.2]. Then the calculation of THH($\mathbb{S}[\Omega Y]$) follows as [1, Corollary IV.3.3]. The highlight of this talk is then [1, Theorem IV.3.6], the calculation of TC($\mathbb{S}[\Omega Y]$) by Bökstedt–Hsiang–Madsen, and its addendum.

Main reference: [1, Chapter IV.3 pp. 112–120.]

Talk 14 (2018 – N.N.): TC *of rings of characteristic p; I.* The first goal is to calculate THH($\mathrm{H}\mathbb{F}_p$), [1, Theorem IV.4.4]. For this, first calculate HH($\mathbb{F}_p$) as in [1, Proposition IV.4.3] and recall something about the Hochschild–Kostant–Rosenberg theorem which is needed for [1, Proposition IV.4.1]. Explain also how to calculate THH($\mathrm{H}\mathbb{F}_p$) using that $\mathrm{H}\mathbb{F}_p$ is a Thom spectrum (and why it is a Thom spectrum). Finish with [1, Proposition IV.4.6].

Main reference: [1, Chapter IV.4 pp. 120–123.]

Talk 15 (2018 – N.N.): TC *of rings of characteristic p; II.* Continue to work towards TC(H$\mathbb{F}_p$). Start with [1, Lemma IV.4.7] and calculate TC(H$\mathbb{F}_p$) which is [1, Corollary IV.4.10]. State [3, Theorem B] to give some context of this calculation to K-theory. Then prove [1, Corollary IV.4.13] and finish with [1, Corollary IV.4.15 & Remark IV.4.16].

Main reference: [1, Chapter II.1 pp. 123–127.]

References

[1] Nikolaus, T., Scholze, P., *On topological cyclic homology.* ArXiv:1707.01799

[2] Nikolaus, T., *Stable ∞-operads and the multiplicative Yoneda lemma.* ArXiv:1608.02901

[3] Hesselholt, L., Madsen, I., *On the K-theory of finite algebras over Witt vectors of perfect fields* Topology 36, 29–101, 1997.