Finite density QCD from an effective lattice theory

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- 3d effective lattice theories derived by strong coupling methods
- The deconfinement transition in Yang-Mills theory  \textit{JHEP 1102 (2011) 057}
- The deconfinement transition in QCD with heavy dynamical quarks \textit{JHEP 1201 (2012) 042}
- Cold and dense QCD: transition to nuclear matter \textit{arXiv:1207.3005}
The (lattice) calculable region of the phase diagram

- Sign problem prohibits direct simulation, circumvented by approximate methods: reweigthing, Taylor expansion, imaginary chem. pot., need $\mu/T \lesssim 1$ ($\mu = \mu_B/3$)
- No critical point in the controllable region, some signals on boundary and beyond
- Flux representations + worm algorithm, complex Langevin: only particular models
Large densities?  Effective theories!

- E.w. phase transition: success with dimensional reduction!
- Scale “separation”: $g^2 T < gT < 2\pi T$
  - Integrate hard scale perturbatively, treat eff. 3d theory on lattice, valid for sufficiently weak coupling
- Does not work for QCD, perturbative dim. red. breaks $Z(3)$ of YM theory
- Bottom-up construction of $Z(N)$-invariant theory by matching:
  - works for SU(2), unfinished for SU(3)
  - Vuorinen, Yaffe; de Forcrand, Kurkela; ... cf. talk Thomas Brauner
- Here: solution for YM by strong coupling expansion!
- QCD, heavy fermions: sign problem of eff. theory milder for reweighting
- Sign problem of eff. theories curable by worm or complex Langevin!
Starting point: Wilson’s lattice Yang-Mills action

\[ Z = \int \prod_{x, \mu} dU(x; \mu) \exp (-S_{YM}) \equiv \int DU \exp (-S_{YM}) \]

Wilson’s gauge action

\[ S_W = -\frac{\beta}{N} \sum_p \text{ReTr}(U_p) = \sum_p S_p \quad \beta = \frac{2N}{g^2} \]

Plaquette:

\[ \square \rightarrow 1 + ia^2 g F_{\mu\nu} - \frac{a^4 g^2}{2} F_{\mu\nu} F^{\mu\nu} + O(a^6) + \ldots \]

\[ U_\mu(x) = e^{-ia g A_\mu(x)} \]

\[ T = \frac{1}{aN_t} \quad \text{continuum limit} \quad a \rightarrow 0, N_t \rightarrow \infty \]

Small \( \beta(a) \Rightarrow \) small \( T \)
The effective theory, Yang-Mills

- Split temporal and spatial link integration and use character expansion \( (a_r(\beta): \text{expansion parameter of representation} \ r) \)

\[
Z = \int [dW] \exp \left\{ \ln \int [dU_i] \prod_p \left[ 1 + \sum_{r \neq 0} d_r a_r(\beta) \chi_r(U_p) \right] \right\}
\]

\[
\equiv \int [dW] \exp [-S_{\text{eff}}] \quad W(\vec{x}) = \prod_{\tau=1}^{N_\tau} U_0(\tau, \vec{x})
\]

Expansion parameter: \( u = a_f(\beta) = \beta/18 + \cdots \)

\[-S_{\text{eff}} = \lambda_1 S_1 + \lambda_2 S_2 + \lambda_3 S_3 + \cdots \]

- \( S_n \) depend only on Polyakov loops
Effective one-coupling theory for SU(3) YM

\( Z = \int [dL] \exp \left[ -S_1 + V_{SU(3)} \right] \)

\[ = \int [dL] \prod_{\langle ij \rangle} \left[ 1 + 2\lambda_1 \text{Re} \left( L_i L_j^* \right) \right] \]

\[ \times \prod_i \sqrt{27 - 18|L_i|^2 + 8\text{Re}L_i^3 - |L_i|^4} \]

\[ \lambda(u, N_\tau \geq 5) = u^{N_\tau} \exp \left[ N_\tau \left( 4u^4 + 12u^5 - 14u^6 - 36u^7 + \frac{295}{2}u^8 + \frac{1851}{10}u^9 + \frac{1055797}{5120}u^{10} \right) \right] \]
Numerical results for SU(3)

Order-disorder transition
First order phase transition for SU(3) in the thermodynamic limit!
Subleading couplings

Subleading contributions for next-to-nearest neighbours:

\[ \lambda_2 S_2 \propto u^{2N_T + 2} \sum_{[kl]}^I 2 \text{Re}(L_k L_l^*) \quad \text{distance} = \sqrt{2} \]

\[ \lambda_3 S_3 \propto u^{2N_T + 6} \sum_{\{mn\}}^I 2 \text{Re}(L_m L_n^*) \quad \text{distance} = 2 \]

as well as terms from loops in the *adjoint* representation:

\[ \lambda_a S_a \propto u^{2N_T} \sum_{<ij>} \text{Tr}^{(a)} W_i \text{Tr}^{(a)} W_j ; \quad \text{Tr}^{(a)} W = |L|^2 - 1 \]
The influence of a second coupling

NLO-couplings: next-to-nearest neighbour, adjoint rep. loops

...gets very small for large \( N_T \)!
Mapping back to 4d finite T Yang-Mills

Inverting

\[ \lambda_1(N_\tau, \beta) \rightarrow \beta_c(\lambda_{1,c}, N_\tau) \]

...points at reasonable convergence
Comparison with 4d Monte Carlo

Relative accuracy for $\beta_c$ compared to the full theory

SU(2)  

SU(3)

Note: influence of additional couplings checked explicitly!
Continuum limit feasible!

- error bars: difference between last two orders in strong coupling exp.
- using non-perturbative beta-function (4d T=0 lattice)
- all data points from one single 3d MC simulation!
Including heavy, dynamical Wilson fermions

Expand in the hopping parameter $\kappa = 1/(2aM + 8)$:

$$S = S_{\text{gauge}} + N_f \sum_{\ell=1}^{\infty} \frac{\kappa^\ell}{\ell} \text{Tr} H[U]^\ell$$

In general the model becomes (with $\overline{h}_i(\mu) = h_i(-\mu)$)

$$-S_{\text{eff}} = \sum_i \lambda_i(u, \kappa, N_T) S_i^S - 2N_f \sum_i \left[ h_i(u, \kappa, \mu, N_T) S_i^A + \overline{h}_i(u, \kappa, \mu, N_T) S_i^{\dagger A} \right]$$

Now, keep only $\lambda_1 S_1^S$ and $h_1 S_1^A + \overline{h}_1 S_1^{\dagger A}$ (now called just $\lambda$, $h$)

Higher powers of loops are resummed into a determinant:

$$Z_{\text{eff}}(\lambda_1, h_1, \overline{h}_1; N_T) = \int [dL] \left( \prod_{\langle ij \rangle} \left[ 1 + 2\lambda_1 \text{Re} L_i L_j^* \right] \right) \left( \prod_x \det[(1 + h_1 W_x)(1 + \overline{h}_1 W_x^\dagger)]^{2N_f} \right) \\
\equiv Q(L_x, L_x^*)^{N_f}$$
QCD: first order deconfinement transition region

Phase diagram in eff. theory:

deconfinement p.t.:
breaking of global $Z(3)$ symmetry;
explicitly broken by quark masses
transition weakens
Phase boundary, numerically

To find $\lambda_{pc}(h)$, $\lambda$-scans were performed at various fixed $h$

- peak in $\chi_O = \langle O^2 \rangle - \langle O \rangle^2$
- dip in $B_O = 1 - \frac{\langle O^4 \rangle}{3 \langle O^2 \rangle^2}$

$$\lambda_{pc}(h) = 0.18805 - 1.797 \cdot h$$
The critical point

\[ \lambda_c = 0.18672(7), \ h_c = 0.000731(40) \]

**Mapping back to QCD:**

\[ e^{-M/T} \approx h/N_f \]

[linear approximation in \( h \ll 1 \ldots \)]

Accuracy \( \sim 5\% \), predictions for \( N_t=6,8,\ldots \) available!

**Convergence properties:**

<table>
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<tr>
<th>( N_f )</th>
<th>( M_c/T )</th>
<th>( \kappa_c(N_\tau = 4) )</th>
<th>( \kappa_c(4), \text{Ref. } [23] )</th>
<th>( \kappa_c(4), \text{Ref. } [22] )</th>
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<tr>
<td>1</td>
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<tr>
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<td>3</td>
<td>8.32(5)</td>
<td>0.0625( 9)</td>
<td>0.0595(3)</td>
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</table>

\( \kappa^4 \) _\text{chiral} \) }
Finite density: sign problem!

- Metropolis algorithm: Mild sign problem; $\frac{\mu}{T} \lesssim 3$
- Worm algorithm: No sign problem  cf. Gattringer et al.

Figure: Quark density calculated with $Z_{\text{eff}}$ from Metropolis or worm algorithm on $24^3$ lattices for $\frac{\mu}{T} = 1$ and 2.
The fully calculated deconfinement transition

\[ \left( \frac{\mu}{T} \right)^2 \approx \begin{cases} 3 & 2nd, Z_2 \\ 2.5 & M_{tric} \approx 6.3 \\ 2 & M_s/T \approx 6.7 \\ 1.5 & M_{tric}/T \approx 5.6 \\ 1 & M_{tric}/T \approx 6.7 \\ 0.5 & \text{tricrit. Roberge-Weiiss} \\ 0 & M_{u,d}/T \end{cases} \]

deconfinement critical surface

phase diagram for Nf=2, Nt=6
Cold and dense QCD I: static, strong coupling limit

For $T=0$ (at finite density) anti-fermions decouple

$$C_f \equiv (2\kappa_f e^{a\mu_f})^{N_\tau} = e^{(\mu_f-m_f)/T}, \quad \tilde{C}_f(\mu_f) = C_f(-\mu_f)$$

$$Z(\beta = 0) = \left[ \prod_f \int dW (1 + C_f L + C_f^2 L^* + C_f^3)^2 \right]^{N_{s^3}}$$

$$\lim_{T \to 0} \left[ 1 + 4C^{N_c} + C^{2N_c} \right]^{N_{s^3}}$$

Free gas of baryons!

Quarkyonic?

$$n = \frac{T}{V} \frac{\partial}{\partial \mu} \ln Z = \frac{1}{a^3} \frac{4N_c C^{N_c} + 2N_c C^{2N_c}}{1 + 4C^{N_c} + C^{2N_c}}$$

$$\lim_{\mu \to \infty} (a^3n) = 2N_c$$

Silver blaze property + saturation!

$$\lim_{T \to 0} a^3n = \begin{cases} 0, & \mu < m \\ 2N_c, & \mu > m \end{cases}$$
Cold and dense QCD II: interacting, “dynamical”!

$Z(3)$ breaking part (fermion determinant), including corrections $\sim \kappa^2$

$m_\pi = 20 \text{ GeV}, T = 10 \text{ MeV}, a = 0.17 \text{ fm}$

$\beta = 5.7, \kappa = 0.0000887, N_\tau = 116$

Complex Langevin: convergence criteria satisfied, cf. Seiler, Stamatescu; Aarts, James

Analytic strong coupling soln. valid!

$\lambda(\beta = 5.7, N_\tau = 115) \sim 10^{-27}$
Continuum extrapolation

Scaling with lattice spacing:

\[
\frac{n_{\text{lat}}(\mu)}{m_B^3} = \frac{n_{\text{cont}}(\mu)}{m_B^3} + A(\mu)a + B(\mu)a^2 + \ldots
\]

Solid/dashed lines: analytic strong coupling limit with/without \( O(\kappa^2) \):

Breakdown of hopping series!
Onset transition to cold nuclear matter

... with very heavy quarks \( m_\pi = 20 \text{ GeV} \)

continuum limit with 5-7 lattice spacings per point

Order of magnitude:

Physical nuclear density

\[ \mu T \approx 4000 \]

\[ \sim 0.16 \text{ fm}^{-3} \approx 0.15 \cdot 10^{-2} \rho_{\text{proton}}^3 \]

Complex Langevin: no sign problem convergence criteria satisfied
cf. Seiler, Stamatescu; Aarts, James
Order of the onset transition?

Distribution of fermion density: 

Reason: exp. short range nucl. potential for heavy pions, \[ V \sim \frac{e^{-m\pi r}}{r} \]
Lighter quarks: First order signal + endpoint!

Preliminary

- $O(k^4)$: Stretching the hopping series, $\kappa = 0.12$, $\beta = 5.6$
- Coexistence of vacuum and finite density phase: 1st order
- If the temperature $T = \frac{1}{aN_T}$ or the quark mass is raised this changes to a crossover
Conclusions

Two-step treatment of QCD phase transitions:

I. Derivation of effective action by strong coupling expansion
II. Simulation of effective theory

Z(N)-invariant effective theory for Yang-Mills, correct order of p.t.,
Tc with better than 10% accuracy in the continuum limit!

Finite T deconf. transition for heavy fermions and all chemical potentials

Silver blaze property + 1st. order phase transition to nuclear matter at T=0
Backup slides
How is this possible?

- $\beta = 0$
  - strong coupling limit
  - radius of convergence
- $\beta = \infty$
  - continuum limit
  - continuum extrapolation
  - scaling region

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Observable to identify order of p.t.:

$$\delta B_Q = B_4(\delta Q) = \frac{\langle (\delta Q)^4 \rangle}{\langle (\delta Q)^2 \rangle^2}$$

$$B_4(x) = 1.604 + b L^{1/\nu} (x - x_c) + \ldots$$
Critical quark mass as function of chemical potential

dense massive limit \( \kappa \to 0, \mu \to \infty, \kappa e^{\mu/T} = \text{constant} \)

- Roberge-Weiss (tricritical) endpoint at \( \mu_i/T = \pi/3 \)
  (\( \leftrightarrow \) boundary between \( Z_3 \)-sectors)

  Tricritical scaling works perfectly, even well into \( \mu^2 > 0 \)!

\[
\frac{M_c}{T} = \frac{M_{\text{tric}}}{T} + K \left[ \left( \frac{\pi}{3} \right)^2 + \left( \frac{\mu}{T} \right)^2 \right]^{2/5}
\]